Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 3, 1999

CORRELATION BETWEEN THEORETICAL AND EXPERIMENTAL SOLITARY WAVES

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UDC 532.59

Experimental data on surface solitary waves generated by five methods are given. These data and literature information show that at amplitudes $0.2 < a/h < 0.6$ *(*h *is the initial depth of the liquid), experimental solitary waves are in good agreement with their theoretical analogs obtained using the complete model of liquid potential flow. Some discrepancy is observed in the range of small amplitudes. The reasons why free solitary waves of theoretically limiting amplitude have not been realized in experiments are discussed, and an example of a forced wave of nearly limiting amplitude is given. The previously established fact that during evolution from the state of rest, undular waves break when the propagation speed of their leading front reaches the limiting speed of propagation of a solitary wave is confirmed.*

In the present work, we consider plane solitary waves on the free surface of a liquid of uniform density and finite depth h above an even horizontal bottom. In the unperturbed state, the liquid is at rest. The information obtained on such waves till 1952 was reviewed by Daily and Stephan [1], who examined 11 theoretical solutions and verified them in detailed experiments. All the solutions considered in [1] referred to waves of small amplitude and differed greatly from each other. This did not permit the notion of a "solitary wave" to be given uniquely, and the fundamental problem of the limiting amplitude a and propagation speed c of a solitary wave remained to be solved. The studies of Daily and Stephan [1] provided the best support for the solutions of [2, 3], which agreed well with experimental data up to $a \approx 0.5h$. In the experiments of [1], the largest wave amplitude reached $0.62h$.

Subsequently, "soliton-like" solutions were obtained in various equations of mathematical physics, in particular, in models for the motion of liquid stratified in density. For surface waves, algorithms that extremely restrict the term of a "solitary wave" and do not contain limitations on its amplitude were proposed, justified, and used in numerical calculations.

In its most limited sense, a theoretical free solitary wave is defined as a solution of the complete equations of liquid potential flow that satisfies the standard (for this model) boundary conditions on the solid boundary and free surface and the conditions of steadiness, symmetry about the wave crest (including at infinity), and monotonic lowering of the free-surface level on each side of the crest [4–8]. The existence theorems for the corresponding solutions are proved by Amick and Toland [6]. At the same time, Longuet-Higgins [5] showed by numerical calculations that these solutions are not unique, and Plotnikov [7] proved this rigorously.

Detailed information on an ideal solitary wave is contained in [4, 5], where it is noted, in particular, that in calculations of the limiting parameters of a solitary wave, one has to introduce *a priori* additional assumptions, which are responsible for the difference between the results of different authors. This difference, however, is not as considerable as the one in the theories verified in the experiments of Daily and Stephan [1]. Among recent algorithms that gave solitary-wave solutions for the complete model of potential flow, the algorithm developed by Ovsyannikov [8] should be mentioned.

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 3, pp. 44–52, May–June, 1999. Original article submitted June 23, 1998; revision submitted October 14, 1998.

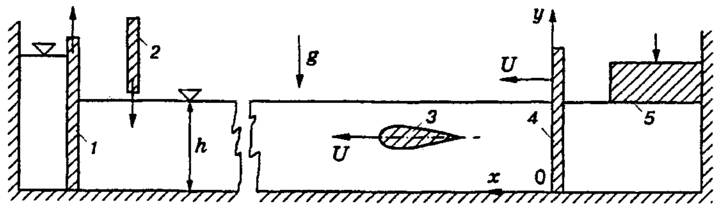

Fig. 1. Diagram of the experiments: 1) baffle; 2) gate; 3) wing; 4) vertical plate; 5) submerged body.

The numerical solution of [5] practically corresponds to an ideal theoretical solitary wave, and it is desirable to verify it in experiments. In particular, Longuet-Higgins and Fenton [5] showed that the solution is not unique and obtained fairly exact limiting amplitude $a_m = 0.827h$ and limiting propagation speed $c_m = 1.294\sqrt{gh}$ for an ideal solitary wave (g is the acceleration of gravity). In addition, they tabulated values of the mass, momentum, energy, and specially determined circulation for an ideal wave.

In experiments, and especially in natural processes, none of the above-mentioned conditions for the existence of an ideal solitary wave is rigorously satisfied. A real wave is always unsteady, the flow generated by it is not potential, and the symmetry conditions ahead of and behind the wave are inevitably violated. In experiments, it is particularly difficult to meet with equal accuracy the necessary conditions on mass, momentum, energy, and circulation for an ideal solitary wave. This makes the problem of correlation between ideal and real solitary waves very important. This problem was analyzed using the above-mentioned works, the experimental information from [9–17], and the results of the experiments described below.

The experiments were performed in a tank 7.3 m long and 0.2 m wide (Fig. 1). The following five methods of generating perturbations were used.

Method 1. This method was used in [1, 13] and other experiments. A certain volume of a liquid was poured into the working section of the tank by short-duration removal of baffle 1. This easily ensures the necessary conditions on the mass of an ideal solitary wave. As regards the other integral conditions, the extent to which they are met depends entirely on the skill of the experimenter [1, 13] since there is just one easily variable parameter of the perturbation — the difference in levels at the baffle — and the other conditions have to be selected by intuitive variation of the law of motion of the baffle. When the baffle is removed too rapidly or too slowly, a solitary wave is generated nevertheless, but it has a rather small amplitude and forms at a great distance from the perturbation source.

Method 2. Baffle 1 was rapidly removed and was not placed again. The initial difference in level at the baffle was such that it ensured obviously excessive values of the mass, momentum, and energy of an ideal solitary wave. This resulted in formation of an unsteady *undular* wave with several crests and troughs. As soon as the first crest of the undular wave became similar to a solitary wave, the tail of the wave was cut off by the rapidly falling gate 2. This method is easier to automate since it contains two easily controllable additional parameters of the perturbation: the distance from the baffle to the gate and the time interval between the rise of the baffle and the lowering of the gate.

Method 3. This method was used, for example, in [9, 14]. Perturbation was induced by horizontal displacement of vertical plate 4. In [9], the plate covered the entire cross section of the tank and prevented overflow of the liquid from above. In these experiments, along with the indicated particular case, the more general case was studied in which the plate incompletely covered the cross section of the tank. This makes another easily variable parameter available to the experimenter. The following law of motion of the vertical plate was specified:

$$U(t) = \begin{cases} U_0\Big[1 - \exp\Big(-\dfrac{t}{T_1}\Big)\Big], & 0 \leqslant t < T_2, \\ 0, & t \geqslant T_2, \end{cases}$$

where t is time and U_0, T_1, and T_2 are parameters. In this case, even at shallow initial water depth in the tank $h = 3$ cm, it was possible to obtain solitary waves that were in good agreement with theoretical waves, which is not easy to realize by removing the baffle [1].

Method 4. This method can be regarded as a more complicated variant of method 3. Perturbation was induced by completely submerged body 3 moving along the tank. In this case, the liquid can flow around the obstacle not only from below, as in incomplete submersion of the plate (see method 3), but also from above, and the experimenter gains another easily variable parameter — the distance of the body to the free surface h_1. The shape of the body and its angle of attack are of significance. In these experiments, we used a streamlined symmetric wing of width D towered at zero angle of attack. In the transverse direction, the wing extended over the entire width of the tank. Only the above-mentioned law of motion was specified.

Method 5. This method was used previously by Russel [15], who was the first to discover solitary waves. Perturbation was induced by submersion of a solid body 5. Two parameters of this perturbation are easily varied. These are the length of the body along the tank and the depth of submersion of its lower edge. The law of motion of the body required to generate solitary waves of rather large amplitude is more difficult to choose.

We also used information obtained when perturbations were induced by the following two methods: by raising a part of the bottom [16] and by a body falling from air into water [17]. Such perturbations are typical of full-scale conditions. The work of Hammack and Segur [16] is of interest because they first calculated the desired law of rise of the bottom theoretically and then reproduced it in experiments. Their main goal was to break an undular type wave at a given point of the tank. Solitary waves were also obtained, but their amplitudes were far from being limiting. In the experiments of [17], most of the perturbation energy dissipated in the complex processes near the body, and the amplitude of the solitary waves generated at a great distance from the place where the body fell was far from being theoretically limiting.

The departure of the free surface from the equilibrium position $\Delta y = (y - h)$ was measured by wavemeters (y is the ordinate of the free surface in the immovable system shown in Fig. 1). The signals from the wavemeters were put on a recorder and a computer. The propagation speed c of a selected point on the wave profile was calculated from the time required for this point to travel between two fixed wavemeters displaced by $\Delta x = (6\text{–}17)h$; smaller values of Δx were specified in the region where the wave profile varied more rapidly. The particular values of c presented below refer to the crest of a solitary wave or to the first crest of a wave of a more complex shape. The departure of this point from the equilibrium position is assumed to be the amplitude a of the experimental wave. The measurement errors for the propagation speeds and amplitudes were estimated (and decreased) from results of repeated measurements under the same conditions. The coefficients of variation in random errors obtained by this method did not exceed 1% for a and 1.5% for c.

Vertical plate 4 and wing 3 were moved by a towing trolley, whose motion was recorded by special slide-wire mechanical-to-electric converters. The measuring system incorporated transducers for synchronizing the readings of the wavemeters with the initial time of perturbation generation. The instrumental measurements were accompanied by visual observations, accomplished, as a rule, by several executers, and, in some experiments, video recording was performed. In particular, the beginning of generation of oblique waves and beginning of breaking of the basic plane wave were detected visually. This information was synchronized in the x coordinate with the readings of the wavemeters. Visually, breaking began with a slide of a small mass of the liquid from the wave crest along its leading edge like an avalanche. A similar mechanism took place at the final stage of the reverse transition of breaking to smooth waves.

The theoretical and experimental waves were compared with respect to the shape of their profile $\Delta y(x)$, the relation between their amplitude and propagation speed, and, in particular cases, in the theoretical relation between the mass and potential energy given in [6]. The reference scales for dimensionless quantities were h and $\sqrt{gh}$. In particular, $a^0 = a/h$ and $c^0 = c/\sqrt{gh}$.

In Fig. 2, selected experimental data are compared to some theoretical results in the plane of parameters (a^0, c^0). The data were obtained for experimental waves that had the shape shown in Fig. 3 and for which the ratio of the mass to the potential energy differed from the theoretical value of [4, 5] by not more than 2%. Experimental points 5 are taken from [1]. Points 6 and 7 are obtained in the present experiments. The first

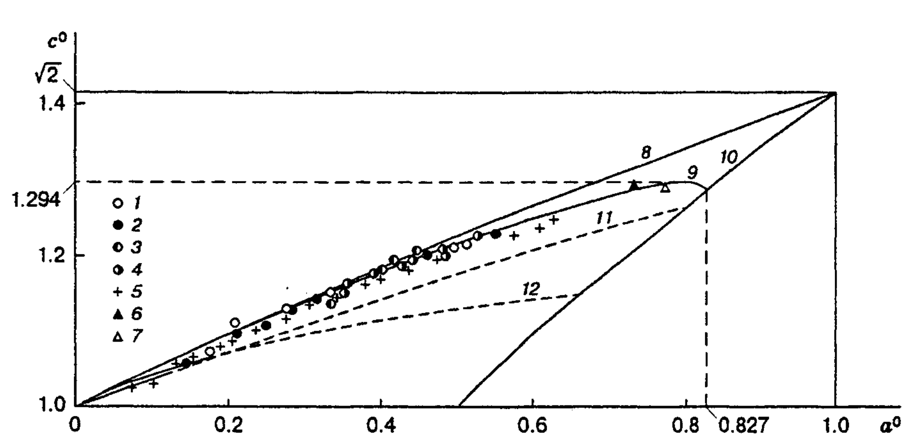

Fig. 2. Comparison of theoretical and experimental amplitudes and propagation speeds: points 1 refer to submersion of a piston, points 2 to motion of a vertical plate, 3 to removal of the baffle, 4 to removal of the baffle and lowering of the gate, 5 are results of [1], 6 are results of [14], points 7 refer to motion of the wing, curve 8 is plotted using the theories of [2, 3], curve 9 is plotted by the calculations in [5], curve 10 is the boundary of the region of existence according to [6], curve 11 is plotted by the of theory [18], and curve 12 is plotted by the theory of [19].

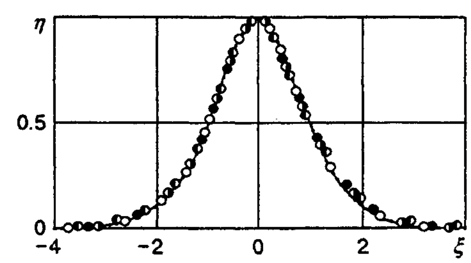

Fig. 3. Profile of a solitary wave generated by different methods (the notation is the same as in Fig. 2).

point refers to a substantially unsteady wave, and the second refers to a *forced* solitary wave. The conditions under which these points were obtained will be discussed below.

The calculated curve 9 from [5] practically corresponds to an ideal wave. The nonuniqueness of the solution was manifested by the fact that in the neighborhood of the limiting amplitude, one value of c^0 corresponds to two values of a^0. Curves 8, 11, and 12 are plotted by the formulas for waves of small amplitude presented in [1]. They demonstrate that until the solutions of the complete equations of liquid potential flow was analyzed, the term "solitary wave" had too broad a sense.

Curve 10 reflects one interesting result of the theory of [6]. On this curve, the speed of liquid particles u_m on the crest of an ideal wave becomes equal to c. If c is treated as the rate of transfer of information on perturbation and u_m is treated as the (local) rate of transfer of mass or energy, then, according to the theory of [6], curve 10 is the boundary of the fundamental physical exclusion that states that the rate of transfer of mass or energy cannot exceed the rate of transfer of information on perturbation. On this boundary, a theoretical solitary wave reaches the limiting amplitude. The experiment shows that the physical exclusion is manifested not only in the behavior of solitary waves but also in other practically important cases. If, during development of a perturbation, a corresponding critical situation arises somewhere in the liquid, the flow pattern changes qualitatively. In the present experiments, waves that appear, during their evolution, to the right of curve 10 invariably break. In the experiments of [17] with a body falling from air on water, even discontinuity of the liquid took place.

It should be noted that by this criterion, the abscissa $c^0 = 1$ in Fig. 2 can also be treated as a critical boundary, but only within the framework of the *linear* theory of waves. On the abscissa, the phase speed of propagation of infinitely long linear harmonic waves becomes equal to their group speed, which characterizes the energy transfer rate. In experiments, with passage through this boundary, oblique waves invariably appear on the fundamental plane wave. In particular, oblique waves are also present on a real solitary wave, and this is an important feature that distinguishes this wave from an ideal wave. In practice, the propagation of oblique waves is inhibited by surface tension. It is not impossible, however, that oblique waves lead to loss of stability before the boundary 10 in Fig. 2 is reached. This is the reason why in experiments a free solitary wave of theoretically limiting amplitude has not been reproduced.

The experimental data obtained with different methods of wave generation agree with each other and with the experimental results of [1] within the measurements errors. The slight systematic deviation of the experimental points in [1] can be explained by the fact that they were obtained for greater distance $\Delta x/h$ between wavemeters than that in the experiments considered. In the class of *free* waves, none of the above methods of generating perturbations gave solitary waves with $a^0 > 0.58$. In [1] and [9], the largest experimental waves had $a^0 \cong 0.62$. However, from the information contained in [1, 9], a quantitative comparison with the theory can be performed only by the relation between the amplitude and propagation speed. At the same time, an important limiting factor is, for example, the marked difference between experimental and ideal waves of large amplitude by such criterion as the ratio of the mass to potential energy.

In both the present experiments and in the experiments of [1], systematic deviation from an ideal wave at $a^0 < 0.2$ took place. From physical considerations, such deviation is reasonable since in real systems, flow rearrangement begins before and terminates after this or that critical situation predicted by the schematic model, and the waves from the indicated range are rather close to the lower boundary of the region of existence of solitary waves.

In the range $0.2 < a^0 < 0.6$, ideal and real waves agree well in the relations between the amplitude and propagation speed and between the mass and potential energy. Here the most significant difference was the asymmetry of the experimental wave shown in Fig. 3, where the experimental profiles of solitary waves with amplitudes in the neighborhood of $a^0 = 0.55 \pm 0.03$ are plotted in the coordinates $\xi = x_1/l_*$ and $\eta = \Delta y/a$ attached to the wave crest. The value x_1 was calculated from the formula $x_1 = x_0 + c(t - t_0)$, where x_0 is the coordinate of the wavemeter in the fixed system and t_0 is the time taken for the wave crest to travel from the coordinate origin to the wavemeter. As l_* we use the value of x_1 for which $\eta = 1/2$. The notation of the experimental points is the same as in Fig. 2. The solid curve shows the symmetric profile $\eta = \mathrm{sech}^2 0.8813\xi$, on which the numerical coefficient is selected empirically from the condition $\eta = 1/2$ at $\xi = \pm 1$. The leading edge of the experimental wave is well approximated by this formula, and the trailing edge is more gently sloping.

The difficulties in generating a solitary wave of theoretically limiting amplitude in experiments are discussed using the data shown in Figs. 4 and 5. Figure 4 shows two sample trajectories in the "phase" plane (a^0, c^0) for the perturbations induced by the motion of a vertical plate. The experimental points A, B, and C correspond to the waves A, B, and C in Fig. 5. The experimental values a^0 and c^0 refer to the first wave crest. During evolution from the state of rest, the perturbations were always to the left of the critical curve 8. The wave profiles at the points A, B, and C on one of the trajectories are given in Fig. 5. At points 1 and 3, the waves were forced since at the corresponding times the plate still moved. The remaining points are obtained after the stop of the plate. At points 5, the leading edge of the wave broke. At the remaining points of the trajectories the waves were smooth.

On the trajectory with experimental points 1 and 2, the waves remained smooth until complete degeneration. At a particular stage of evolution, a solitary wave formed, which is in good agreement with the results of [2, 5], but by this time its amplitude became considerably smaller than a^0_m. It is interesting that during further degeneration due to viscosity and reflection from the end walls of the tank, the parameters of the experimental solitary wave were similar to the theoretical values for an ideal liquid for a long time. That is, the following quasisteadiness principle was satisfied: as the amplitude decreased, the propagation speed decreased accordingly.

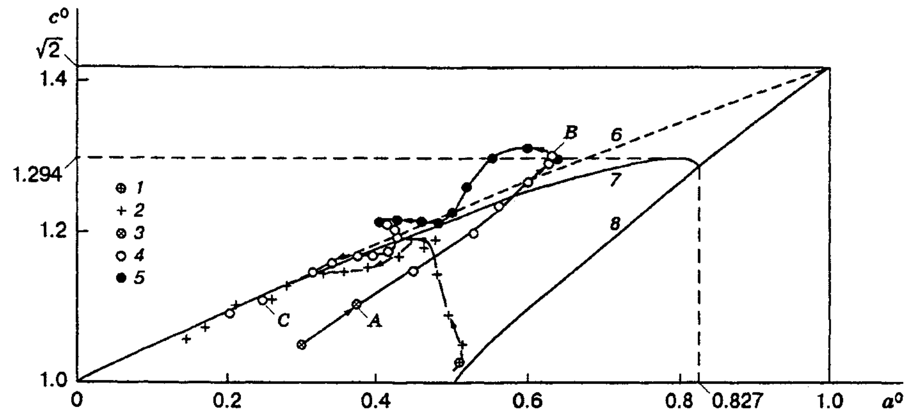

Fig. 4. Examples of trajectories of real processes in the plane (a^0, c^0): points 1 and 3 refer to forced smooth waves, points 2 and 4 to free smooth waves, points 5 to free breaking waves, curve 6 shows the results of [2, 3], curve 7 shows the results of [5], and curve 8 is plotted from the data of [6].

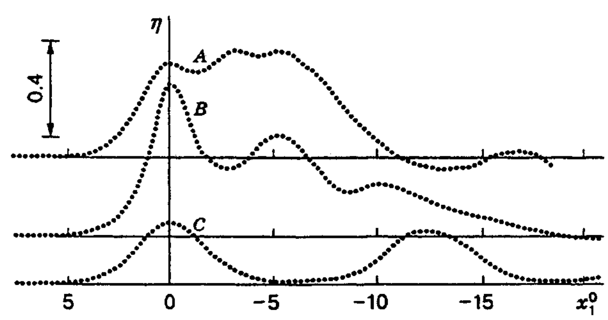

Fig. 5. Breakup of an initial complex perturbation into solitary waves.

In the above example, the fluid mass set in motion by the plate was smaller than the theoretical mass for a limiting solitary wave. In the example with experimental points 3–5, the initial perturbation had an obviously excessive mass. As a result, breaking of the leading edge began at the point B of the perturbation trajectory. It is interesting that although the wave at this point of the trajectory differed greatly in shape from a solitary wave (see Fig. 5) and was substantially unsteady, it began to break when its propagation speed was the same as the maximum propagation speed of an ideal solitary wave. More detailed information on this issue is contained in [14]. The reverse transition from breaking to smooth waves occurred at lower propagation speed $c^0 \approx 1.22$. At the point C of the trajectory considered, two solitary waves (see Fig. 5) were generated. During further degeneration, the parameters of the first wave traced the corresponding values for an ideal wave.

The data in Fig. 4 show that in order for a theoretical solitary wave of limiting amplitude to be produced in experiments, the trajectory of the corresponding process must reach the point of intersection of curves 7 and 8, remaining all the time to the left of curve 8, and the perturbation should have strictly definite mass, momentum, and energy. Up to not, it has not been possible to meet these requirements even in laboratory experiments.

Experimental waves that are similar to solitary waves and have a nearly theoretical amplitude a_m are quite often obtained in *unsteady* and *forced* processes. Such a wave was recorded, for example, in the experiments of [14] (point 6 in Fig. 2). It remained smooth for a much shorter time than the other experimental points and then broke.

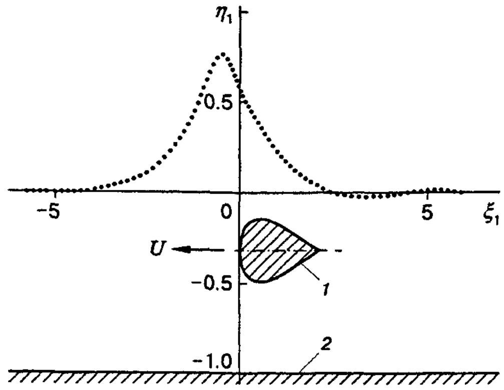

Fig. 6. Forced solitary wave generated by a wing moving at the critical velocity: 1) wing; 2) bottom of the tank; $U/c_m = 0.99 \pm 0.01$, $D/h = 1/3$, and $h_1/h = 1/3$.

A forced wave of large amplitude (point 7 in Fig. 2) was obtained in the present experiments by generating perturbations by a moving wing. This wave is shown in Fig. 6. Here $\xi_1 = x_2/h$; x_2 differs from the previously determined x_1 only in that the reference point is attached not to the wave crest but to the leading edge of the wing (in Fig. 6, the shape of the wing is distorted because of different scales on the coordinate axes; the real wing had an elongation of 6 : 1). The ordinate is $\eta_1 = \Delta y/h$. The wave was steady while the wing moved at constant speed. When the wing stopped, it broke rapidly into waves with a sign-alternating deviation from the equilibrium position.

The wave in Fig. 6 is of interest since it forms only for a strictly definite combinations of the parameters of the system and perturbation, i.e., it is parametrically unstable; small variations of the parameters of the system or perturbation lead to a strong qualitative change in the wave pattern. A necessary conditions for the existence of the wave considered is the equality $U = c_m$ ($U =$ const is the wing speed). In the experiments performed at $U/c_m = 0.97$, a train of smooth waves similar to solitary waves propagated unrestrictedly far upstream. The number of separate waves in the train continuously increased, and their amplitudes reached $0.6a_m$. At $U/c_m > 1$, the perturbations ahead of the body were absent or they propagated in the form of breaking waves. This depended on the combination of other parameters, for example, the degree of flow blocking by the wing or the depth of location of the wing under the free surface.

Thus, the theory of an ideal solitary wave adequately describes real objects, at least, in the range $0.2 < a^0 < 0.6$. The range $0.6 < a^0 < a^0_m$ has not been adequately investigated in experiments so far. It can be noted only that *free* solitary waves from this range can hardly be generated in natural processes since even in a laboratory experiment, it is not possible to meet all the necessary conditions for their existence. For an adequate description of real waves with $a^0 < 0.2$, it is necessary to use more complex mathematical models than the model of potential flow. A particular important result of the theory of ideal solitary waves is that it revealed the critical propagation speed $c_m \cong 1.294\sqrt{gh}$, which appears in different problems, for example, in propagation of undular and forced nonlinear waves.

The author thanks A. V. Gusev, I. N. Potapov, and N. P. Turanov for the considerable contribution to the derivation of the initial experimental data.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 98-01-00750) and the Foundation for the Integration Programs of the Siberian Division of the Russian Academy of Sciences (Grant No. 97-43).

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